Journal of Analysis and Applications
Vol. 21 (2023), No.2, pp.89-99
ISSN: 0972-5954

URL : www.sasip.net

Some properties of Fibonacci-sigmoid numbers and polynomials matrix

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Abstract. In this paper, we introduce the Fibonacci-sigmoid polynomials $S_{n,F}(x)$ and the Fibonacci-sigmoid matrix $\boldsymbol{\mathcal{S}}_{n,F}(x,F)$. Also, examples of the inverse of the Fibonacci-sigmoid matrix are presented. We factorize the following matrix by the Fibonacci matrix.

AMS Subject Classification (2020): 11B39, 15A16, 68T07

Keywords: Fibonacci exponential function, Fibonacci-sigmoid polynomials, symmetric property

1. Introduction

Many mathematicians have recently studied various matrices for different types of polynomials and sequences. For instance, Pascal's matrices were studied in depth by Gregory S. Call and Daniel J. Velleman (see [2], [10]).

In addition, Fibonacci-Pascal matrices and the inverse of these matrices were studied and organized (see [4], [9]). Furthermore, other matrices such as Bernoulli and Euler matrices were studied extensively as well.

In this study, we focus mainly on the matrices which contain entries regarding Fibonacci-sigmoid polynomials. In order to do so, we first define a basic exponential function which includes the concept of Fibonacci numbers.

Definition 1.1 [5]. The Fibonacci exponential function e_F^t is defined as

$$e_F^t = \sum_{n=0}^{\infty} \frac{t^n}{F_n!}.$$

Definition 1.2 [5], [8]. The Fibonacci sequence $\{F_n\}_{n\geq 0}$ is defined by

$$F_n = \begin{cases} F_{n+2} = F_{n+1} + F_n, \\ F_0 = 0, F_1 = 1. \end{cases}$$

Definition 1.3 [5], [8]. For $1 \leq n \leq m$, the Fibonacci coefficients are defined by

$$\binom{m}{n}_F = \frac{F_m!}{F_{m-n}!F_n!},$$

where $F_m! = F_m F_{m-1} F_{m-2} \cdots F_1$, $F_0! = 1$.

We note $\binom{m}{0}_F = 1$ and $\binom{m}{n}_F = 0$ for $m < n$.

Definition 1.4 [1], [3], [7]. The sigmoid numbers and polynomials are defined by

$$\sum_{n=0}^{\infty} S_n \frac{t^n}{n!} = \frac{1}{e^{-t}+1},$$

$$\sum_{n=0}^{\infty} S_n(x) \frac{t^n}{n!} = \frac{1}{e^{-t}+1} e^{tx},$$

respectively.

Hence, we introduced the basic definition of the Fibonacci sequence and the Fibonomial coefficients. The exponential function which includes Fibonacci numbers were also established.

Furthermore, in order to instigate a relation between Fibonacci sequence and the Pascal triangle, the Fibonacci-Pascal matrix and its' inverse were defined. Finally, the sigmoid numbers and polynomials expressed via the generating function was shown.

The paper is outlined as follows:

In Section 2, we define the Fibonacci-sigmoid polynomials and numbers to show their relation. In Section 3, we introduce the Fibonacci-sigmoid polynomials matrix and calculate an example of the matrix. Then, we check if the additivity property holds for the matrix.

We also calculate several of the inverse matrices. Finally, the Fibonacci-sigmoid polynomials matrices can be factorized by the Fibonacci matrix by defining a new matrix.

2. Some properties for Fibonacci-sigmoid numbers and polynomials

In this section, we define new Fibonacci-sigmoid polynomials and derive properties of these numbers and polynomials. We also find several symmetric identities for Fibonacci-sigmoid polynomials.

Definition 2.1. Let n be a non-negative integer. Then, we define Fibonacci sigmoid polynomials as

$$\sum_{n=0}^{\infty} S_{n,F}(x)\frac{t^n}{F_n!} = \frac{1}{e_F^{-t}+1}e_F^{tx}.$$

For $x = 0$ in Definition 2.1, we note

$$\sum_{n=0}^{\infty} S_{n,F}\frac{t^n}{F_n!} = \frac{1}{e^{-t}+1}.$$

where we call $S_{n,F}$ the Fibonacci sigmoid numbers.

We have a relation $S_{n,F}(x) = E_{n,F}(-x)$, where $E_{n,F}(x)$ is the Fibonacchi Euler polynomials (see [6]).

Theorem 2.2. *Let k be a non-negative integer. Then, we have*

$$S_{n,F}(x) = \sum_{k=0}^{n} \binom{n}{k}_F x^k S_{n-k,F}.$$

Proof. Using Fibonacci-sigmoid numbers in the generating function of Fibonacci- sigmoid polynomials, we have

$$\begin{aligned}
&\sum_{n=0}^{\infty} S_{n,F}(x)\frac{t^n}{F_n!}\\
&= \frac{1}{e_F^{-t}+1}e_F^{tx}\\
&= \sum_{n=0}^{\infty} S_{n,F}\frac{t^n}{F_n!}\sum_{n=0}^{\infty} x^n\frac{t^n}{F_n!}\\
&= \sum_{n=0}^{\infty}\left(\sum_{k=0}^{n}\binom{n}{k}_F x^k S_{n-k,F}\right)\frac{t^n}{F_n!}. \qquad (1)
\end{aligned}$$

Comparing the coefficients of the both-sides in Equation (1), we complete the proof of Theorem 2.1. □

Theorem 2.3. *Let $x, y \in \mathbb{C}$. Then, the following equation*

$$S_{n,F}(x+y) = \sum_{k=0}^{n}\binom{n}{k}_F y^k S_{n-k,F}(x)$$

holds.

Proof. Replacing $x + y$ instead of x in the generating function of the Fibonacci- sigmoid polynomials, we have

$$\sum_{n=0}^{\infty} S_{n,F}(x+y)\frac{t^n}{F_n!} = \frac{1}{e_F^{-t}+1}e_F^{t(x+y)}. \qquad (2)$$

Using the Definition 2.1 in the right-hand side of Equation (2), we obtain

$$\begin{aligned}
&\sum_{n=0}^{\infty} S_{n,F}(x+y)\frac{t^n}{F_n!}\\
&= \sum_{n=0}^{\infty}\left(\sum_{k=0}^{n}\binom{n}{k}_F S_{n-k}(x)y^k\right)\frac{t^n}{F_n!}, \qquad (3)
\end{aligned}$$

which is the required result using comparison of coefficients. □

Corollary 2.4. *Consider $y = 1$ in Theorem 2.3. Then, we have the complement property as*

$$S_{n,F}(1+x) = \sum_{k=0}^{n} \binom{n}{k}_F S_{n-k,F}(x).$$

Theorem 2.5. *Let $\alpha, \beta \neq 0$ and α, β be non-negative integers. Then, we derive*

$$\sum_{k=0}^{n} \alpha^{n-k}\beta^{k} S_{n-k}(\alpha^{-1}x) S_k(\beta^{-1}y)$$
$$= \sum_{k=0}^{n} \beta^{n-k}\alpha^{k} S_{n-k}(\beta^{-1}x)(\alpha^{-1}y).$$

Proof. Suppose A is defined as the following form:

$$A := \frac{1}{(e_F^{-\alpha t}+1)(e_F^{-\beta t}+1)} e_F^{t(x+y)}. \tag{4}$$

Applying the generating function of Fibonacci-sigmoid polynomials in Equation (4), we obtain

$$A := \sum_{n=0}^{\infty} S_n(\alpha^{-1}x)\frac{(\alpha t)^n}{F_n!} \sum_{n=0}^{\infty} S_n(\beta^{-1}y)\frac{(\beta t)^n}{F_n!}$$
$$= \sum_{n=0}^{\infty}\left(\sum_{k=0}^{n} \alpha^{n-k}\beta^{k} S_{n-k}(\alpha^{-1}x;u) S_k(\beta^{-1}y;u)\right)\frac{t^n}{F_n!}, \tag{5}$$

and

$$A := \frac{1}{e_F^{-\beta t}+1} e_F^{tx} \frac{1}{e_F^{-\alpha t}+1} e_F^{ty}$$
$$= \sum_{n=0}^{\infty} S_n(\beta^{-1}x)\frac{(\beta t)^n}{F_n!} \sum_{n=0}^{\infty} S_n(\alpha^{-1}y)\frac{(\alpha t)^n}{F_n!}$$
$$= \sum_{n=0}^{\infty}\left(\sum_{k=0}^{n} \beta^{n-k}\alpha^{k} S_{n-k}(\beta^{-1}x) S_k(\alpha^{-1}y)\right)\frac{t^n}{F_n!}. \tag{6}$$

From Equations (5) and (6), we obtain the basic symmetric relation within the Fibonacci-Sigmoid polynomials and finish the proof of Theorem 2.5. □

Corollary 2.6. *For $\alpha = 1$ in Theorem 2.5, we have*

$$\sum_{k=0}^{n} \beta^k S_{n-k}(x) S_k(\beta^{-1}y) = \sum_{k=0}^{n} \beta^{n-k} S_{n-k}(\beta^{-1}x) S_k(y).$$

Corollary 2.7. *Let $x = 0$ in Theorem 2.5. Then, we have another symmetric property as*

$$\sum_{k=0}^{n} \alpha^{n-k}\beta^k S_{n-k} S_k(\beta^{-1}y) = \sum_{k=0}^{n} \beta^{n-k}\alpha^k S_{n-k}(\alpha^{-1}y).$$

Theorem 2.8. *Let k, n be non-negative integers. Then, we find*

$$\sum_{k=0}^{n} \binom{n}{k}_F (-1)^{n-k} S_{k,F} + S_{n,F} := \begin{cases} 1, & \text{if } \quad n = 0, \\ 0, & \text{if } \quad n > 0. \end{cases}$$

Proof. We suppose $e_F^{-t} \neq -1$ in the generating function of the Fibonacci-sigmoid numbers. From the generating function of these numbers and e_F, we find

$$\sum_{n=0}^{\infty} S_n \frac{t^n}{F_n!} \left(\sum_{n=0}^{\infty} (-1)^n \frac{t^n}{F_n!} + 1 \right) = 1. \tag{7}$$

From Equation (7), we get

$$\sum_{n=0}^{\infty} \left(\sum_{k=0}^{n} \binom{n}{k}_F (-1)^{n-k} S_k + S_n \right) \frac{t^n}{F_n!} = 1. \tag{8}$$

Using Equation (8), we obtain the desired result. □

Corollary 2.9. *Using the similar method of proof from Theorem 2.8, we hold*

$$\sum_{k=0}^{n} \binom{n}{k}_F (-1)^{n-k} S_{k,F}(x) + S_{n,F}(x) = x^n.$$

3. Fibonacci-sigmoid polynomials matrix

This section contains the results regarding the matrices of the Fibonacci-sigmoid numbers and polynomials.

We begin by defining the matrices of the Fibonacci-sigmoid polynomials and exhibit an example of the matrix. Furthermore, additivity property and its' generalized formula is shown for the Fibonacci-sigmoid polynomials matrix.

Finally, we define the inverse matrix of the Fibonacci-sigmoid polynomials matrix and give several examples of the matrix..

Definition 3.1. Let $S_{n,F}(x)$ be the $n-1^{th}$ Fibonacci-sigmoid polynomial.

The $(n+1)\times(n+1)$ Fibonacci-sigmoid polynomials matrix is

$$\boldsymbol{\mathcal{S}}_{n,F}(x,F) = [s_{ij}(x,F)] \text{ for } i,j = 1,2,...,n$$

is defined by

$$s_{ij}(x,F) = \begin{cases} \binom{i}{j}_F S_{i-j,F}(x), & if\ i \geq j, \\ 0, & if\ otherwise. \end{cases}$$

We note the Fibonacci-sigmoid number matrix for $x = 0$ as $\boldsymbol{\mathcal{S}}_n(0,F) = \boldsymbol{\mathcal{S}}_n(F)$.

Example 3.2. Consider $n = 3$ for the Fibonacci-sigmoid polynomials matrix. We find

$$\boldsymbol{\mathcal{S}}_3(x,F) = \begin{bmatrix} S_{0,F}(x) & 0 & 0 & 0 \\ S_{1,F}(x) & S_{0,F}(x) & 0 & 0 \\ S_{2,F}(x) & S_{1,F}(x) & S_{0,F}(x) & 0 \\ S_{3,F}(x) & 2S_{2,F}(x) & 2S_{1,F}(x) & S_{0,F}(x) \end{bmatrix}$$

$$= \begin{bmatrix} \frac{1}{2} & 0 & 0 & 0 \\ \frac{1}{2}x + \frac{1}{4} & \frac{1}{2} & 0 & 0 \\ \frac{1}{2}x^2 - \frac{x}{4} - \frac{3}{8} & \frac{1}{2}x + \frac{1}{4} & \frac{1}{2} & 0 \\ \frac{1}{2}x^3 + \frac{1}{2}x^2 - \frac{3}{8}x - \frac{5}{8} & 2\left(\frac{1}{2}x^2 - \frac{x}{4} - \frac{3}{8}\right) & 2\left(\frac{1}{2}x + \frac{1}{4}\right) & \frac{1}{2} \end{bmatrix}.$$

Theorem 3.3. *Additivity property regarding variables x and y hold for $\boldsymbol{\mathcal{S}}_n(x,F)$ as follows:*

$$\boldsymbol{\mathcal{S}}_n(x+y,F) = \boldsymbol{\mathcal{S}}_n(x,F)\boldsymbol{\mathcal{S}}_n(y,F) = \boldsymbol{\mathcal{S}}_n(y,F)\boldsymbol{\mathcal{S}}_n(x,F).$$

Proof. For $i > j$, using the previous definition, we obtain

$$\begin{aligned} s_{ij}(x+y,F) &= \binom{i}{j}_F S_{i-j,F}(x+y) \\ &= \sum_{k=0}^{i-j} \binom{i}{j}_F \binom{i-j}{k}_F S_{i-j-k,F}(x)S_{k,F}(y). \end{aligned} \tag{9}$$

For $k \geq j$, the above equation (9) changes as follows:

$$\begin{aligned} s_{ij}(x+y,F) &= \sum_{k=j}^{i} \binom{i}{j}_F \binom{i-j}{k-j}_F S_{i-k,F}(x)S_{k-j,F}(y) \\ &= s_{ij}(x,F)s_{ij}(y,F). \end{aligned}$$

Replacing x and y, it is proved. □

Corollary 3.4. *Generalizing Theorem 3.3, the following holds.*

Let $(x_1, x_2, ..., x_k) \in \mathbb{R}^k$. The matrices $\boldsymbol{\mathcal{S}}_n(x_j)$ for $j = 0, 1, ..., k$ satisfies the following product formula:

$$\boldsymbol{\mathcal{S}}_n(x_1 + x_2 + ... + x_k, F) = \boldsymbol{\mathcal{S}}_n(x_1,F)\boldsymbol{\mathcal{S}}_n(x_2,F)...\boldsymbol{\mathcal{S}}_n(x_k,F).$$

Theorem 3.5. *$\boldsymbol{\mathcal{S}}_{n,F}(x,F)$ satisfies the following formulae:*

$$\begin{aligned} &\boldsymbol{\mathcal{S}}_{n,F}(x+y,F) \\ &= \boldsymbol{\mathcal{P}}_n[x,F]\boldsymbol{\mathcal{S}}_{n,F}(y,F) \\ &= \boldsymbol{\mathcal{P}}_n[y,F]\boldsymbol{\mathcal{S}}_{n,F}(x,F). \end{aligned}$$

If $y = 0$, $\boldsymbol{\mathcal{S}}_{n,F}(x,F) = \boldsymbol{\mathcal{P}}_{n,F}[x,F]\boldsymbol{\mathcal{S}}_{n,F}(F)$ holds as well.

Proof. For $i \geq j$, we have

$$\begin{aligned}
s_n(x+y,F) &= \binom{i}{j}_F S_{i-j,F}(x+y)\\
&= \binom{i}{j}_F \sum_{k=0}^{i-j} \binom{i-j}{k}_F S_{k,F}(y)x^{i-j-k}\\
&= \sum_{k=j}^{i} \binom{i}{j}_F \binom{i-j}{k-j}_F S_{k-j,F}(y)x^{i-k}\\
&= \sum_{k=j}^{i} \binom{i}{k}_F x^{i-k} \binom{k}{j}_F S_{k-j,F}(y)\\
&= p_n(x;i,k)s_{k,j}(y,F).
\end{aligned}$$

Hence, it is proved. □

Example 3.6. Let $\boldsymbol{\mathcal{D}}_{n,F}(x) \in M_{n+1}(\mathbb{R})$ be the inverse matrix of the Fibonacci-sigmoid polynomials matrix $\boldsymbol{\mathcal{S}}_{n,F}(x)$. Then, several inverse Fibonacci-sigmoid polynomials matrices are as follows.

(i) If $n = 1$, by calculating the determinant of $\boldsymbol{\mathcal{S}}_{n,F}(x)$, we can easily find

$$\boldsymbol{\mathcal{D}}_{1,F}(F) = \begin{bmatrix} 2 & 0\\ -2x-1 & 2 \end{bmatrix}.$$

(ii) If $n = 2$, by using the Gauss–Jordan elimination, we can find the inverse matrix

$$\boldsymbol{\mathcal{D}}_{2,F}(F) = \begin{bmatrix} 2 & 0 & 0\\ -2x-1 & 2 & 0\\ 3x+2 & -2x-1 & 2 \end{bmatrix}.$$

(iii) If $n = 3$, by using the same method, we find

$$\boldsymbol{\mathcal{D}}_{3,F}(x) = \begin{bmatrix} 2 & 0 & 0 & 0\\ -2x+1 & 2 & 0 & 0\\ 3x+2 & -2x-1 & 2 & 0\\ 2x^3-8x^2-\frac{25}{2}x-2 & 6x+4 & -4x-2 & 2 \end{bmatrix}.$$

4. Conclusion

Based on the given results in Fibonacci-Euler polynomials matrix (see [6]), we defined the Fibonacci-sigmoid polynomials matrix and organized several properties related to this matrix. It would be useful later on to expand the sigmoid polynomial to other polynomials and define a more generalized matrix in order to find more common properties.

Acknowledgement. The author would like to express her sincere gratitude to the referees for their valuable suggestions and comments which improved the paper.

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(*Received: May, 2023; Revised: June, 2023*)